Heyting Algebras and Kripke Models for Intuitionistic Logic

Michael Barkasi
mtb2@rice.edu

March 2011

1 Introduction

The following are notes I prepared for an hour guest lecture in advanced logic at Rice University in the Spring of 2011. Most of the material is drawn from Dirk van Dalen's article "Intuitionistic Logic" (2002), Carnielli et al *Analysis and Synthesis of Logic* (2008), Steve Awodey, *Category Theory* (2006), and Olaf Beyersdorff and Oliver Kutz's lecture notes from ESSLLI 2010 on proof complexity of non-classical logics.

2 Familiar Territory: CPC

2.1 Boolean Algebras

A *Boolean algebra* consists of a set $\mathcal{B}$ which contains (at least) the two elements $\top$ ("top") and $\bot$ ("bottom") and the operations $\sqcap$ ("meet"), $\sqcup$ ("join") and $-$ ("complement") such that for all $x, y \in \mathcal{B}$,

1. $\sqcap$ and $\sqcup$ are commutative and associative
2. $(x \sqcap y) \sqcup y = y$ and $(x \sqcup y) \sqcap y = y$
3. $\sqcap$ is distributive w.r.t. $\sqcup$ and vice versa
4. $-(-x) = x$
5. $-(x \sqcup y) = (-x) \sqcap (-y)$ and $-(x \sqcap y) = (-x) \sqcup (-y)$
6. $x \sqcap \top = x$ and $\bot \sqcap x = \bot$
7. $x \sqcup (-x) = \top$ and $x \cap (-x) = \bot$

When no confusion arises we denote a Boolean algebra by its underlying set $\mathcal{B}$, although it's properly specficed by the whole tuple

$$(\mathcal{B}, \sqcap, \sqcup, -, \top, \bot)$$

We can also give Boolean algebras special names when convenient, as we do below. Note that we can define a partial ordering $\leq$ on $\mathcal{B}$ by setting

$$x \leq y \Leftrightarrow x \sqcap y = x$$

Clearly top $\top$ is maximal w.r.t. this ordering, while bottom $\bot$ is minimal.

2.2 Examples

The Boolean algebra giving the standard semantics for **CPC** is something like

$$(\{t, f\}, \wedge, \vee, \neg, t, f)$$

where $\wedge$, $\vee$ and $\neg$ are the respective operations on $\{t, f\}$ defined by the usual truth tables. We'll denote this Boolean algebra by *2*. Let X be a set and $\mathcal{P}(X)$ its power set. Then

$$(\mathcal{P}(X), \cap, \cup, /, X, \varnothing)$$

is a Boolean algebra. The ordering relation induced on the Boolean algebra $\mathcal{P}(X)$ turns out to be subset inclusion $\subseteq$. Let *ats* be some set of propositional atoms such as $\{p_1, p_2, p_2, \ldots\}$ and let L be the set of all formulas of **CPC** generated over *ats* by the connectives $\wedge$, $\vee$, $\neg$, $\supset$ and $\equiv$. For $\varphi, \psi \in L$, define an equivalence relation

$$\varphi \sim \psi \Leftrightarrow \vdash_{\mathbf{CPC}} \varphi \equiv \psi$$

Write $[\varphi]_\sim$ for the equivalence class of φ. Then if $L/\sim = \{[\varphi]_\sim : \varphi \in L\}$ is the set of equivalence classes,

$$(L/\sim, \wedge, \vee, \neg, [p_1 \vee \neg p_1]_\sim, [p_1 \wedge \neg p_1]_\sim)$$

is a Boolean algebra, called the *Lindenbaum algebra*. The operations are defined in the obvious way: $[\varphi]_\sim \wedge [\psi]_\sim = [\varphi \wedge \psi]_\sim$, $[\varphi]_\sim \vee [\psi]_\sim = [\varphi \vee \psi]_\sim$ and $\neg[\varphi]_\sim = [\neg\varphi]_\sim$. Denote the Lindenbaum algebra of **CPC** by $\mathcal{L}_{\mathbf{CPC}}$.

2.3 Basic Results

A *valuation* of L in $(\mathcal{B}, \sqcap, \sqcup, -, \top, \bot)$ is given by an assignment $v : ats \to \mathcal{B}$ which is extended to all of L as follows:

1. $v(\varphi \wedge \psi) = v(\varphi) \sqcap v(\psi)$
2. $v(\varphi \vee \psi) = v(\varphi) \sqcup v(\psi)$
3. $v(\neg\varphi) = -v(\varphi)$
4. $v(\varphi \supset \psi) = v(\neg\varphi \vee \psi)$
5. $v(\varphi \equiv \psi) = v((\varphi \supset \psi) \wedge (\psi \supset \varphi))$

A formula $\varphi \in L$ is *valid* in $\mathcal{B}$ if and only if $v(\varphi) = \top$ for all valuations v of L in $\mathcal{B}$.

Theorem:
$\vdash_{\mathbf{CPC}} \varphi \Leftrightarrow \varphi$ is valid in *2*.

Proof:
Found in any logic text.

Theorem:
$\vdash_{\mathbf{CPC}} \varphi \Leftrightarrow \varphi$ is valid in every Boolean algebra $\mathcal{B}$.

Proof:
Soundness, i.e. that $\vdash_{\mathbf{CPC}} \varphi \Rightarrow \varphi$ is valid in every Boolean algebra $\mathcal{B}$, is easily checked.

Completeness, i.e. that $\vdash_{\mathbf{CPC}} \varphi \Leftarrow \varphi$ is valid in every Boolean algebra $\mathcal{B}$, can be seen by considering that if φ is valid in every Boolean algebra, then in particular it's valid in $\mathcal{L}_{\mathbf{CPC}}$. Hence every valuation of L in $\mathcal{L}_{\mathbf{CPC}}$ is s.t. $v(\varphi) = [p_1 \vee \neg p_1]_{\sim}$. In particular consider the valuation which sends all formulas ψ to the element $[\psi]_{\sim}$. So $[\varphi]_{\sim} = [p_1 \vee \neg p_1]_{\sim}$. By definition of $\sim$, $\vdash_{\mathbf{CPC}} \varphi \equiv (p_1 \vee \neg p_1)$. Clearly any proof of $\varphi \equiv (p_1 \vee \neg p_1)$ can be extended to a proof of φ.

An alternative proof of completeness is that if φ is valid in every Boolean algebra, then in particular it's valid in *2*. But by the last theorem, this implies that $\vdash_{\mathbf{CPC}} \varphi$.

3 Heyting Algebras and IPC

3.1 Heyting Algebras

A *Heyting Algebra* consists of a set $\mathcal{H}$ which contains (at least) the two elements $\top$ ("top") and $\bot$ ("bottom") and the operations $\sqcap$ ("meet"), $\sqcup$ ("join") and $\to$ ("exponential") such that for all $x, y, z \in \mathcal{H}$,

1. $\sqcap$ and $\sqcup$ are commutative and associative
2. $(x \sqcap y) \sqcup y = y$ and $(x \sqcup y) \sqcap y = y$
3. $x \to (y \sqcap z) = (x \to y) \sqcap (x \to z)$
4. $x \sqcap (x \to y) = x \sqcap y$ and $(x \to y) \sqcap y = y$
5. $(x \to x) \sqcap y = y$
6. $x \sqcap \top = x$ and $\bot \sqcap x = \bot$

Again a partial ordering can be defined on a Heyting Algebra so that $x \leq y \Leftrightarrow x \sqcap y = x$. Every Boolean algebra forms a Heyting algebra by defining $x \to y = (-x) \sqcup y$. Besides Boolean algebras, the most natural examples of Heyting algebras are topological spaces: if X is a set and $\mathcal{O}_X$ a topology on X, then

$$(\mathcal{O}_X, \cap, \cup, \to, X, \varnothing)$$

where $U \to V = ((X/U) \cup V)^\circ$, for $U, V \subseteq X$ open (i.e. $U, V \in \mathcal{O}_X$,) is a Heyting algebra.[1] Another example of a Heyting algebra is the *Lindenbaum algebra* for **IPC**: it is constructed over L exactly as with $\mathcal{L}_{\mathbf{CPC}}$, except this time $\varphi \sim \psi \Leftrightarrow \varphi$ and ψ are logically equivalent in **IPC** and we have to pick different representatives for top and bottom, say $p_1 \supset p_1$ and $\neg(p_1 \supset p_1)$. We denote this Heyting algebra by $\mathcal{L}_{\mathbf{IPC}}$.

To see that not every Heyting algebra is Boolean, consider that if complement $-$ is to be defined on a Heyting algebra it must be $-x = x \to \bot$. In a Boolean algebra $x \sqcup -x = \top$, but this isn't necessarly so in a Heyting algebra. When $\mathcal{H}$ is a topological space $\mathcal{O}_X$ and $U, V \in \mathcal{O}_X$, $-U = (X/U)^\circ$, $U \sqcup V = U \cup V$ and $\top = X$. So $U \sqcup -U = U \cup (X/U)^\circ$, which isn't necessarly X. Consider the following counter example.

[1] For $Y \subseteq X$, Y° is the interior of Y, i.e. the union of all open sets $U \in \mathcal{O}_X$ contained in Y.

The standard topology $\mathcal{O}_{\mathbb{R}}$ on $\mathbb{R}$ can be given as follows. For $x, r \in \mathbb{R}$, let $B_{x,r} = \{y \in \mathbb{R} : |x - y| < r\}$. Then $\mathcal{O}_{\mathbb{R}}$ is the smallest subset of $\mathcal{P}(\mathbb{R})$ s.t. $\mathbb{R}, \varnothing \in \mathcal{O}_{\mathbb{R}}$, $B_{x,r} \in \mathcal{O}_{\mathbb{R}}$ for all x, r, the union of any collection of sets in $\mathcal{O}_{\mathbb{R}}$ is itself in $\mathcal{O}_{\mathbb{R}}$ and the intersection of any two sets in $\mathcal{O}_{\mathbb{R}}$ is itself in $\mathcal{O}_{\mathbb{R}}$. Now note that $\{x : 0 < x\} \in \mathcal{O}_{\mathbb{R}}$. If we call this set U, $-U = \{x : 0 > x\}$, so $U \cup -U = \mathbb{R}/\{0\} \neq \mathbb{R}$.

3.2 Basic Results

As before, a *valuation* of L in $(\mathcal{H}, \sqcap, \sqcup, \rightarrow, \top, \bot)$ is given by an assignment $v : ats \rightarrow \mathcal{H}$ which is extended to all of L as follows:

1. $v(\varphi \wedge \psi) = v(\varphi) \sqcap v(\psi)$
2. $v(\varphi \vee \psi) = v(\varphi) \sqcup v(\psi)$
3. $v(\neg\varphi) = -v(\varphi) = v(\varphi) \rightarrow \bot$
4. $v(\varphi \supset \psi) = v(\varphi) \rightarrow v(\psi)$
5. $v(\varphi \equiv \psi) = v((\varphi \supset \psi) \wedge (\psi \supset \varphi))$

A formula $\varphi \in L$ is *valid* in $\mathcal{H}$ if and only if $v(\varphi) = \top$ for all valuations v of L in $\mathcal{H}$.

Theorem:
$\vdash_{\mathbf{IPC}} \varphi \Leftrightarrow \varphi$ is valid in $\mathcal{O}_{\mathbb{R}}$.

Theorem:
$\vdash_{\mathbf{IPC}} \varphi \Leftrightarrow \varphi$ is valid in every Heyting algebra $\mathcal{H}$.

Theorem:
There does not exist a single Heyting algebra $\mathcal{H}$ with only finitely many elements s.t. $\vdash_{\mathbf{IPC}} \varphi \Leftrightarrow \varphi$ is valid in $\mathcal{H}$.

Proof of Theorem 2.2:
As before, soundness can be shown by comparing the axioms for Heyting algebras against a proof system for **IPC**. Completeness is again proved via the Lindenbaum algebra: if φ is valid in all Heyting algebras, it's valid in $\mathcal{L}_{\mathbf{IPC}}$. So in particular the valuation v sending formulas ψ to $[\psi]_\sim$ is s.t. $v(\varphi) = [p_1 \supset p_1]_\sim$, i.e. $[\varphi]_\sim = [p_1 \supset p_1]_\sim$, so the proof of $p_1 \supset p_1$ in **IPC** can be easily extended to a proof of φ.

4 Heyting Algebras and Kripke Semantics

4.1 Review of Kripke Semantics

Recall that a *Kripke model* M is given by a set W, the "worlds" or "points" or "states", a relation $R \subseteq W \times W$ which specifies which worlds are "accessible" from which, and a function $V : W \to \mathcal{P}(ats)$ which specifies which atomic formulas are true in each world. We usually write $M = (W, R, V)$. The usual definition for satisfaction in a Kripke model is as follows:

1. $M, w \models_{\mathbf{Grz}} p \Leftrightarrow p \in V(w)$
2. $M, w \models_{\mathbf{Grz}} \varphi \wedge \psi \Leftrightarrow M, w \models_{\mathbf{Grz}} \varphi$ and $M, w \models_{\mathbf{Grz}} \psi$
3. $M, w \models_{\mathbf{Grz}} \varphi \vee \psi \Leftrightarrow M, w \models_{\mathbf{Grz}} \varphi$ or $M, w \models_{\mathbf{Grz}} \psi$
4. $M, w \models_{\mathbf{Grz}} \neg\varphi \Leftrightarrow M, w \not\models_{\mathbf{Grz}} \varphi$
5. $M, w \models_{\mathbf{Grz}} \varphi \supset \psi \Leftrightarrow M, w \models_{\mathbf{Grz}} \varphi$ implies $M, w \models_{\mathbf{Grz}} \psi$
6. $M, w \models_{\mathbf{Grz}} \varphi \equiv \psi \Leftrightarrow M, w \models_{\mathbf{Grz}} (\varphi \supset \psi) \wedge (\psi \supset \varphi)$

If $M, w \models_{\mathbf{Grz}} \varphi$ for all $w \in W$ we say that φ is "true" or *valid* in M and write $M \models_{\mathbf{Grz}} \varphi$.

Also recall that if we restrict our attention to Kripke models M s.t. R is a partial order—i.e. is reflective, transitive and antisymmetric—and assignments V on W s.t. $wRw' \Rightarrow V(w) \subseteq V(w')$ and we modify the satisfaction relation we get a sound and complete semantics for **IPC**. Let us write $\leq$ for the accessibility relation of such Kripke models and denote the worlds by lower-case greek letters $\alpha, \beta, \ldots$ Define:

1. $M, \alpha \models_{\mathbf{IPC}} p \Leftrightarrow p \in V(\alpha)$
2. $M, \alpha \models_{\mathbf{IPC}} \varphi \wedge \psi \Leftrightarrow M, \alpha \models_{\mathbf{IPC}} \varphi$ and $M, \alpha \models_{\mathbf{IPC}} \psi$
3. $M, \alpha \models_{\mathbf{IPC}} \varphi \vee \psi \Leftrightarrow M, \alpha \models_{\mathbf{IPC}} \varphi$ or $M, \alpha \models_{\mathbf{IPC}} \psi$
4. $M, \alpha \models_{\mathbf{IPC}} \neg\varphi \Leftrightarrow$ for all $\beta \geq \alpha$, $M, \beta \not\models_{\mathbf{IPC}} \varphi$
5. $M, \alpha \models_{\mathbf{IPC}} \varphi \supset \psi \Leftrightarrow$ for all $\beta \geq \alpha$, $M, \beta \models_{\mathbf{IPC}} \varphi$ implies $M, \beta \models_{\mathbf{IPC}} \psi$.

6. $M, \alpha \models_{\mathbf{IPC}} \varphi \equiv \psi \Leftrightarrow M, \alpha \models_{\mathbf{IPC}} (\varphi \supset \psi) \wedge (\psi \supset \varphi)$

The intuitive interpretation of these semantics for **IPC** is that the "states" α represent moments in time and $V(\alpha)$ is "what's known" at α. $\beta \geq \alpha$ if and only if β is a later time. As we move from one time α to a later time β "what's known" always increases, i.e. $V(\alpha) \subseteq V(\beta)$, and we can think of theorems being proven as we progress through time. Then, for example, $\neg\varphi$ holds at a time α just in case it's impossible to ever establish a proof of φ, i.e. if φ is never proven at a later time.

Let us call a Kripke model M whose accessibility relation is a partial order and whose valuation is monotone over that order, i.e. $\alpha \leq \beta \Rightarrow V(\alpha) \subseteq V(\beta)$, a *Heyting model*. Then we have the following completeness theorem:

Theorem:
$\models_{\mathbf{IPC}} \varphi \Leftrightarrow M \models_{\mathbf{IPC}} \varphi$ for all Heyting models M.

4.2 *Heyting Algebras from Heyting Frames*

First some more terminology: those Kripke frames (W, R) s.t. R is a partial order we'll call *Heyting frames*. We'll use H to denote arbitrary Heyting frames. We can "view" a Heyting frame H as a Heyting algebra $\mathcal{H} = T(H)$. *Qua* interpretation of formula, H and the corresponding $T(H)$ are closely related.

Let $H = (W, \leq)$ be a Heyting frame. Then there is a natural topology we can associate with W: $\mathcal{O}_W$ consists of the subsets U of W which are closed upwards on $\leq$, i.e. if $\alpha \in U$ and $\beta \geq \alpha$, then $\beta \in U$. Note that for $\alpha \in W$, the sets $U_\alpha = \{\beta : \beta \geq \alpha\}$ form a basis for $\mathcal{O}_W$. We then set $T(H) = (\mathcal{O}_W, \cap, \cup, \rightarrow, W, \varnothing)$ where $\rightarrow$ is defined as above. (So T is a function from Heyting frames to Heyting algebras.)

Theorem:

1. For each Heyting model $M = (W, \leq, V)$ on a Heyting frame H there is a valuation v_V on L in $T(H)$ s.t. $v_V(\varphi) = \{\alpha : M, \alpha \models_{\mathbf{IPC}} \varphi\}$.

2. Similarly, for each valuation v on L in $T(H)$ there is a Heyting model $M = (W, \leq, V_v)$ on the Heyting frame H such that $V_v(\alpha) = \{p \in ats : \alpha \in v(p)\}$.

Proof:

To show (1), note that the assignment V of M is a function $W \to \mathcal{P}(ats)$ which is monotone over the ordering $\leq$ of M. It induces a function $v_V : ats \to \mathcal{H}$ by setting $v_V(p) = \{\alpha \in W : p \in V(\alpha)\}$. By induction on the construction of formulas it can be shown that the extension of this v_V to all of L yields a valuation on L in $T(H)$ s.t. $v_V(\varphi) = \{\alpha : M, \alpha \models_{\mathbf{IPC}} \varphi\}$. Note that (2) is obvious, simply consider the valuation V_v so defined.

Corollary:

φ is valid in $T(H) \Leftrightarrow M \models_{\mathbf{IPC}} \varphi$ for all M on H.

Proof:

($\Rightarrow$) Assume that there exists some $M = (W, \leq, V)$ on H s.t. $M \not\models_{I} \varphi$. Then there exists some $\alpha \in W$ s.t. $M, \alpha \not\models_{\mathbf{IPC}} \varphi$. Hence there exists some valuation v_V on L in $T(H)$ s.t. $\alpha \notin v_V(\varphi)$. Hence $v_V(\varphi) \neq W$, so φ is not valid in $T(H)$.

($\Leftarrow$) Similar to before, i.e. we want to prove the contrapositive, but this time we must proceed by induction over the construction of φ. The details get a bit gritty, so I leave it to the reader.

As a last comment, I'm pretty sure—although not positive—that if H is the cannonical frame for **Grz**, then $T(H)$ is isomorphic to $\mathcal{L}_{\mathbf{IPC}}$ as a Heyting algebra. At least, if there is any justice in the world this is so!

5 Translations

5.1 *Semantic Systems*

The above connection between Heyting algebras and Heyting frames can be set in a more general framework.[2] Let L be a language, not necessarily the language of **CPC** and **IPC** described above. Let X be a set and let $\models$ be a binary relation on $X \times L$. We call $(X, L, \models)$ a *semantic system* or something similarly suggestive. The relation $\models$ of a semantic system induces a *consequence relation* between sets Γ of formulas from L and formulas φ from L by setting $\Gamma \models \varphi \Leftrightarrow$ for all $x \in X$, $x \models \gamma$ for all $\gamma \in \Gamma \Rightarrow x \models \varphi$. One may want to restrict semantic systems by requiring that their induced consequence relation satisfy some properties, e.g. $\varphi \in \Gamma \Rightarrow \Gamma \models \varphi$.

[2] For more information on the topics of this section, see Mossakowski et al. 2009.

5.2 Translations

A *translation* $(f_1, f_2) : S_1 \to S_2$ between semantic systems $S_1 = (X_1, L_1, \models_1)$ and $S_2 = (X_2, L_2, \models_2)$ is a pair of functions $f_1 : L_1 \to L_2$ and $f_2 : X_2 \to X_1$ s.t. for all $x \in X_2$, $\varphi \in L_1$:

$$f_2(x) \models_1 \varphi \Leftrightarrow x \models_2 f_1(\varphi)$$

If we let **HF** be the set of all Heyting frames, **HA** be the set of all Heyting algebras and write $\mathcal{H} \models_{alg} \varphi$ when φ is valid in $\mathcal{H}$ and $H \models_{frm} \varphi$ when $M \models_{\mathbf{IPC}} \varphi$ for all M on H, then $\mathrm{Int}_{alg} = (\mathbf{HA}, L, \models_{alg})$ and $\mathrm{Int}_{frm} = (\mathbf{HF}, L, \models_{frm})$ are semantic systems with $(\mathrm{id}, T) : \mathrm{Int}_{alg} \to \mathrm{Int}_{frm}$ as a translation. That is, if $\mathrm{id} : L \to L$ is the identity function mapping formulas of **IPC** to themselves and T is the function defined above sending Heyting frames to Heyting algebras,

$$T(H) \models_{alg} \varphi \Leftrightarrow H \models_{frm} \mathrm{id}(\varphi) \Leftrightarrow H \models_{frm} \varphi$$

5.3 ***IPC*** and the Logic of Proofs

As sketched above, the Heyting model interpretation of intuitionistic logic is suppose to reflect the fact that intuitionistic logic is the "logic of provability" or that intuitionistic logic is "epistemic" in some sense. This intuitive idea can be made precise by providing a formal translation between intuitionistic logic and **Grz**, Grzegorczyk logic, the modal logic where the box operator $\Box$ is meant to be read "there is a proof."

Let L_m be the language of propositional modal logic and $f : L \to L_m$ be the Gödel-McKinsey-Tarski mapping which is defined inductively by

1. $f(p) = \Box p$
2. $f(\varphi \wedge \psi) = f(\varphi) \wedge f(\psi)$
3. $f(\varphi \vee \psi) = f(\varphi) \vee f(\psi)$
4. $f(\neg\varphi) = \Box\neg f(\varphi)$
5. $f(\varphi \supset \psi) = \Box(f(\varphi) \supset f(\psi))$

If **HM** is the set of Heyting models, then if we consider the semantic systems $\mathrm{Int} = (\mathbf{HM}, L, \models_{\mathbf{IPC}})$ and $\mathrm{Prov} = (\mathbf{HM}, L_m, \models_{\mathbf{Grz}})$ we have a translation $(f, \mathrm{id}) : \mathrm{Int} \to \mathrm{Prov}$. That is,

$$\mathrm{id}(M) \models_{\mathbf{IPC}} \varphi \Leftrightarrow M \models_{\mathbf{Grz}} f(\varphi) \Leftrightarrow M \models_{\mathbf{IPC}} \varphi$$

It further turns out that there is a connection with **S4**, that if φ is a formula of **IPC**, then φ is (intuitionistically) valid in all Heyting models if and only if it's valid, on the usual definition of satisfaction, in all reflective and transitive Kripke models.

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