AN EXPOSITION OF MORLEY'S THEOREM

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Contents

1. Introduction

1.1. **Background.** The Löwenheim-Skolem theorem was the first significant result of model theory. It says that if Σ is a signature of cardinality λ and $\Gamma \subseteq L_{\omega\omega}(\Sigma)$ a set of sentences with an infinite model, then Γ has infinite models of cardinality κ for all $\kappa \geq \lambda$. Morley's theorem was one of the first important extentions of Löwenheim-Skolem. It says that if in addition Γ has only one model up to isomorphism of

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cardinality λ for some uncountable λ, then it has only one model up to isomorphism for all uncountable λ. The classical example of such a set Γ is the axioms for algebraically closed fields where we also include a sentence fixing the characteristic of the field. It is well known that the algebraic closure of a field is unique up to isomorphism. [Lan02, Theorem V.2.8, Corollary V.2.9] It turns out, in addition, that for each integer n there is, up to isomorphism, only one algebraically closed field of cardinality λ with $\mathrm{Char} = n$, for all uncountable λ.

Michael Morley's seminal 1965 paper [Mor65] is still easy to read and accessible. But, some of Morley's most important concepts—those for *type* and *rank*—do not take the form one will find in a modern text on model theory. Furthermore, because the theorem requires a lot of theoretical tools to prove, modern texts often have the pieces of the proof scattered throughout various chapters. My goal here is to present a compact treatment of the proof that gives the big picture and connects the definitions Morley used with the more usual ones given in contemporary literature.

1.2. **Notation.** Ordinals are thought of as von Neumann ordinals, i.e. sets of all preceding ordinals. Arbitrary ordinals are denoted by κ, λ, δ, etc, while ω_0, or usually just ω, denotes the ordinal containing all finite ordinals. Cardinal numbers are identified with the smallest ordinal of that cardinality, i.e. using the usual notation for ordinals, $\aleph_i = \omega_i$.[1] Finite ordinals—natural numbers—are denoted by m, n, etc. $x = \infty$ means $x = \lambda$ for all ordinals λ. The terms 'tuple' and 'sequence' are used interchangebly. We often aberivate finite tuples $x_1, \ldots, x_n$ and infinite tuples $(x_i)_{i<\kappa}$ by $\bar{x}$. The powerset of a set X is denoted $\mathcal{P}(X)$, while the cardinality of X is denoted by $\#X$, e.g. $\#\mathbb{N} = \aleph_0$. A quirk of our convention of identifying cardinal numbers with ordinals is that $\#\omega = \omega$, or in general $\#\omega_i = \omega_i$. I use the word 'set' as many would use the word 'class,' but try to avoid contexts where this is problematic.[2]

Calligraphic uppercase letters $\mathcal{A}, \mathcal{B}$, etc will denote structures. The signature of a structure $\mathcal{A}$, written $\sigma(\mathcal{A})$, is the set of function, relation and constant symbols naming the functions, relations and constants of $\mathcal{A}$. If $\mathcal{A}$ is a Σ-structure, i.e. $\sigma(\mathcal{A}) = \Sigma$, and F, R, and c are respectively function, relation and constant symbols of Σ, then $F^{\mathcal{A}}$, $R^{\mathcal{A}}$, and $c^{\mathcal{A}}$ are respectively the function, relation and constant of $\mathcal{A}$ named by them.

[1] The index i here, of course, ranges over all ordinals: i.e. by the Axiom of Choice, there's no reason we can't well-order the cardinal numbers, so writing $\aleph_\kappa$ for some ordinal κ makes sense.

[2] Foundational issues regarding the sizes of collections or other paradoxes of set theory are not relevant to Morley's theorem.

If $\mathcal{A}$ is a substructure of $\mathcal{B}$ we write $\mathcal{A} \leq \mathcal{B}$. If $\mathcal{A}$ is a structure, $|\mathcal{A}|$ is the domain of $\mathcal{A}$.[3] If $X \subseteq |\mathcal{A}|$, $\langle X \rangle$ is the substructure of $\mathcal{A}$ generated by X.[4]

If Σ is a signture, $L_{\omega\omega}(\Sigma)$ is the first-order language of Σ.[5] If Γ is a set of formula, $\sigma(\Gamma)$ is the signature of Γ, i.e. the set of function, relation and constant terms appearing in the formulas of Γ. (When $\Gamma = \{\varphi\}$ for some single formula φ, we write $\sigma(\varphi)$.) If t is a term of $L_{\omega\omega}(\Sigma)$ containing variables $x_1, \ldots, x_n$, it is written $t(x_1, \ldots, x_n)$. Let $t(\bar{x})$ be a term of $L_{\omega\omega}(\Sigma)$ and $\Sigma \subseteq \sigma(\mathcal{A})$, if $\bar{a}$ is a tuple of elements of $\mathcal{A}$ at least as long as $\bar{x}$, then $t^{\mathcal{A}}(\bar{a})$ is the element named by $t(\bar{x})$ when a_i is substituted for x_i. If φ is a formula of $L_{\omega\omega}(\Sigma)$ whose free variables are among $x_1, \ldots, x_n$, it is written $\varphi(x_1, \ldots, x_n)$. Let $\varphi(\bar{x}) \in L_{\omega\omega}(\Sigma)$ and $\Sigma \subseteq \sigma(\mathcal{A})$, if $\bar{a}$ is a tuple of elements of $\mathcal{A}$ at least as long as $\bar{x}$ and $\mathcal{A}$ satisfies $\varphi(\bar{x})$ when a_i is substituted for x_i, we write $\mathcal{A} \models \varphi(\bar{a})$. If φ is a sentence, i.e. a formula without free variables, we simply write $\mathcal{A} \models \varphi$ when $\mathcal{A}$ satifies φ; by definition, if φ is a sentence and $\mathcal{A} \models \varphi$, then $\mathcal{A} \models \varphi(\bar{a})$ for all $\bar{a}$.

Given structures $\mathcal{A}$ and $\mathcal{B}$ such that $\sigma(\mathcal{A}) = \sigma(\mathcal{B})$ and a signature $\Sigma \subseteq \sigma(\mathcal{A})$, a function $f : |\mathcal{A}| \to |\mathcal{B}|$ is a Σ-homomorphism if it preserves all the functions, relations and constants of $\mathcal{A}$ named by Σ.[6] A Σ-embedding is an injective Σ-homomorphism such that f^{-1} preserves all the relations of $\mathcal{B}$ named by Σ.[7] A Σ-isomorphism is a surjective Σ-embedding. When $\Sigma = \sigma(\mathcal{A})$ and $f : |\mathcal{A}| \to |\mathcal{B}|$ is a Σ-homomorphism (resp. Σ-embedding) we simply call it a homomorphism (resp. embedding) and write $f : \mathcal{A} \to \mathcal{B}$ (resp. $f : \mathcal{A} \rightarrowtail \mathcal{B}$).

If $\mathcal{A} \models \varphi$ we say that $\mathcal{A}$ is a model of the sentence φ. The set of all models of a sentence φ is $\mathrm{Mod}(\varphi)$. If Γ is a set of sentences, $\mathrm{Mod}(\Gamma) = \bigcap_{\gamma \in \Gamma} \mathrm{Mod}(\gamma)$. We define $\Gamma \models \varphi$ if and only if $\mathrm{Mod}(\Gamma) \subseteq \mathrm{Mod}(\varphi)$. Two sentences φ, ψ are equivalent if $\mathrm{Mod}(\varphi) = \mathrm{Mod}(\psi)$.[8]

[3] As is made precise below, for any signature Σ, $|\cdot|$ can be thought of as the forgetful functor from the category of Σ-structures to **Set**.

[4] Of course, if $X \subseteq |\mathcal{A}|$ we could consider the sub-Σ-structure of $\mathcal{A}$ generated by X for any $\Sigma \subseteq \sigma(\mathcal{A})$, but by substructure of $\mathcal{A}$ we shall always mean sub-$\sigma(\mathcal{A})$-structure.

[5] In infinitary logic, $L_{\kappa\lambda}(\Sigma)$ is the language generated on Σ with at most $< \kappa$ variables quantified over and at most $< \lambda$ formulas joined in conjunctions/disjunctions.

[6] That is, if (i) $f(F^{\mathcal{A}}(\bar{a})) = F^{\mathcal{B}}(f(\bar{a}))$, (ii) $\bar{a} \in R^{\mathcal{A}} \Rightarrow f(\bar{a}) \in R^{\mathcal{B}}$, and (iii) $f(c^{\mathcal{A}}) = c^{\mathcal{B}}$.

[7] That is, $\bar{b} \in R^{\mathcal{B}} \Rightarrow f^{-1}(\bar{b}) \in R^{\mathcal{A}}$.

[8] Note that this definition is independent of the signatures involved, so long as $\mathrm{Mod}(\varphi)$ is defined as all the Σ-structures satisfying φ for $\Sigma \supseteq \sigma(\varphi)$.

Two formula φ, ψ are equivalent modulo Γ for some set of sentences Γ if $\Gamma \models \forall \bar{x}(\varphi \leftrightarrow \psi)$. The equivalence class of φ modulo Γ is written $[\varphi]_\Gamma$. The set of sentences $\varphi \in L_{\omega\omega}(\Sigma)$ satisfied by some structure $\mathcal{A}$ such that $\Sigma \subseteq \sigma(\mathcal{A})$ is denoted $\mathrm{Th}(\Sigma, \mathcal{A})$. Two structures $\mathcal{A}$, $\mathcal{B}$ such that $\sigma(\mathcal{A}) = \sigma(\mathcal{B})$ are (first-order) Σ-equivalent, where $\Sigma \subseteq \sigma(\mathcal{A})$, written $\mathcal{A} \equiv_\Sigma \mathcal{B}$, if $\mathrm{Th}(\Sigma, \mathcal{A}) = \mathrm{Th}(\Sigma, \mathcal{B})$; $\mathcal{A} \equiv \mathcal{B}$ means they are $\sigma(\mathcal{A})$-equivalent. If $\mathcal{A}$, $\mathcal{B}$ are such that $\sigma(\mathcal{A}) = \sigma(\mathcal{B})$ and are Σ-isomorphic for some $\Sigma \subseteq \sigma(\mathcal{A})$, we write $\mathcal{A} \cong_\Sigma \mathcal{B}$; $\mathcal{A} \cong \mathcal{B}$ means $\mathcal{A} \cong_{\sigma(\mathcal{A})} \mathcal{B}$.

Given a Σ-structure $\mathcal{A}$ and two sets B, C indexed into tuples $\bar{b}$, $\bar{c}$ by the same index set I where $B \subseteq |\mathcal{A}|$, then the C expansion of Σ, written $\Sigma(C)$ or $\Sigma(\bar{c})$, is the signature got by adding the elements of C as constant symbols to Σ. The B expansion of $\mathcal{A}$, denoted $(\mathcal{A}; \bar{b})$ or $(\mathcal{A}; B)$, is $\mathcal{A}$ viewed as a $\Sigma(C)$-structure by setting $c_i^{(\mathcal{A};B)} = b_i$.[9] Let $\mathcal{A}$, $\mathcal{B}$ be structures such that $\sigma(\mathcal{B}) = \sigma(\mathcal{A})$ and $|\mathcal{A}| \subseteq |\mathcal{B}|$, then $\mathcal{B}$ is an elementary extension of $\mathcal{A}$, or $\mathcal{A}$ is an elementary substructure of $\mathcal{B}$, written $\mathcal{A} \preccurlyeq \mathcal{B}$, if the inclusion map $i : |\mathcal{A}| \hookrightarrow |\mathcal{B}|$ preserves all formula, i.e. for all $\varphi \in L_{\omega\omega}(\sigma(\mathcal{A}))$, $\bar{a}$ a tuple of elements from $\mathcal{A}$, $\mathcal{A} \models \varphi(\bar{a})$ if and only if $\mathcal{B} \models \varphi(i(\bar{a}))$. Let $\varphi(x_1, \ldots, x_n) \in L_{\omega\omega}(\Sigma)$ and $\mathcal{A}$ be a structure such that $\Sigma \subseteq \sigma(\mathcal{A})$, then $\varphi(x_1, \ldots, x_n)$ defines a set of tuples of elements of $\mathcal{A}$, i.e. the set $\{(a_1, \ldots, a_n) \in |\mathcal{A}|^n : \mathcal{A} \models \varphi(a_1, \ldots, a_n)\}$. The relation defined by some formula $\varphi(\bar{x})$ in $\mathcal{A}$ is denoted $\varphi(\mathcal{A}^n)$. Further, we say that a relation is definable with parameters from some subset $A \subseteq |\mathcal{A}|$ if it is $\varphi(\mathcal{A}^n, \bar{a})$ for some formula $\varphi(\bar{x}, \bar{y})$ and elements $\bar{a}$ from A.

Finally, a set Γ of sentences is called a theory if it is consistent, i.e. if $\mathrm{Mod}(\Gamma) \neq \varnothing$. A theory T is complete if any two of its models are $\sigma(\mathrm{T})$-equivalent, or alternatively: T is complete if for all sentences $\varphi \in L_{\omega\omega}(\sigma(\mathrm{T}))$, either $\mathrm{T} \models \varphi$ or $\mathrm{T} \models \neg\varphi$. A theory T is categorical if every model of T is $\sigma(\mathrm{T})$-isomorphic. T is λ-categorical if every infinite model of T of infinite cardinality λ is Σ-isomorphic. T is totally categorical if T is λ-categorical for every infinite λ. We can thus state Morley's theorem as follows:

Theorem 1.2.1 (Morley's Theorem)**.** *Suppose T is a first-order theory such that $\sigma(T)$ is countable and that it is λ-categorical for some $\lambda > \omega$, then T is λ-categorical for all $\lambda > \omega$.*

1.3. **Caveats.** This paper started as a term paper for a graduate level algebra course at Rice University. Overall the exposition is neither

[9]Throughout this paper we will normally just let $C = B$, so the signature is expanded by adding the elements themselves as constant symbols.

rigorous nor complete, but hopefully it will be a good guide. Almost nothing of substance below is original to myself—that is, neither the definitions nor theorems originated with me—although I have written the proofs and prepared the presentation. I have drawn heavily from Morley's own work, Hodges's excellect text [Hod97] and David Marker's article [Mar00], as well as several other cited sources.

Indeed, this paper is incomplete in two senses: first it assumes the reader knows the basics of model theory. One should be acquainted with all the concepts mentioned in the above remarks on notation. Second, I have not yet finished writing it. There are five or six proofs that still need to be written. Please email any comments, questions or criticism to mbark551@live.kutztown.edu.

2. Basic Definitions

The proof of Morley's theorem will follow directly from three lemmas. The lemmas require defining the notions of type, saturation and stability. One will find more than a few different, but equivalent, definitions of a type in the literature.

2.1. **Types.**

Roughly, we can think of compete n-types as all the things true of a tuple in a structure that can be said with "parameters." [Hod97, 130] Let $\mathcal{A}$ be a Σ-structure, $B \subseteq |\mathcal{A}|$ the parameters, $\bar{a}$ an n-tuple of elements from $\mathcal{A}$ and $\bar{x}$ an n-tuple of variables. Then:

Definition 2.1.1. $tp_{\mathcal{A}}(\bar{a}/B)$ is the set of all formula $\varphi(\bar{x}) \in L_{\omega\omega}(\Sigma(B))$ such that $(\mathcal{A}; B) \models \varphi(\bar{a})$.

Now let $F_n(\Sigma)$ be the set of formulas in $L_{\omega\omega}(\Sigma)$ with no free variables other than $x_1, \ldots, x_n$. Note that if T is a theory and $\Sigma \supseteq \sigma(T)$, $B_n(\Sigma) = \{[\varphi]_{\mathrm{T}} : \varphi \in F_n(\Sigma)\}$ is a Boolean algebra where

$$\begin{aligned} [\varphi]_{\mathrm{T}} \cdot [\psi]_{\mathrm{T}} &:= [\varphi \wedge \psi]_{\mathrm{T}} \\ [\varphi]_{\mathrm{T}} + [\psi]_{\mathrm{T}} &:= [\varphi \vee \psi]_{\mathrm{T}} \\ -[\varphi]_{\mathrm{T}} &:= [\neg\varphi]_{\mathrm{T}} \\ 1 &:= [x_1 = x_1]_{\mathrm{T}} \\ 0 &:= [\neg x_1 = x_1]_{\mathrm{T}} \end{aligned}$$

If $\mathcal{A}$ is a Σ-structure, the set of all closed atomic formulas and negated closed atomic formulas of $\Sigma(|\mathcal{A}|)$ which are satisfied by $\mathcal{A}$ is called the *diagram* of $\mathcal{A}$ and denoted by $D(\mathcal{A})$.

Definition 2.1.2. Let $\mathcal{A}$ be a Σ-structure, $B \subseteq |\mathcal{A}|$. The following are equivalent and define a *n-type of $\mathcal{A}$ over B*.[10] Let $p \subseteq F_n(\Sigma(B))$.

(1) for every finite subset $p' \subseteq p$, there exists an n-tuple $\bar{a}$ of elements of $\mathcal{A}$ such that $\mathcal{A} \models \varphi(\bar{a})$ for all $\varphi \in p'$.
(2) $p \subseteq tp_{\mathcal{A}'}(\bar{a}/B)$ for some n-tuple $\bar{a}$ of elements of $\mathcal{A}'$, $\mathcal{A} \preccurlyeq \mathcal{A}'$.
(3) $[p]_{\mathrm{T}} = \{[\varphi]_{\mathrm{T}} \in B_n(\Sigma(B)) : \varphi \in p\}$ is a filter of $B_n(\Sigma(B))$, for $\mathrm{T} = \mathrm{Th}(\Sigma, \mathcal{A}) \cup D(\langle B \rangle)$

Note that $p = tp_{\mathcal{A}'}(\bar{a}/B)$ if and only if $[p]_{\mathrm{T}}$ is an ultrafilter if and only if for all $\varphi \in F_n(\Sigma(B))$, either $\varphi \in p$ or $\neg\varphi \in p$. If any of these hold we say that p is a *complete n-type of $\mathcal{A}$ over B*. We say the elements $\bar{a}$ *realize* the complete n-type p if $p \subseteq tp_{\mathcal{A}}(\bar{a}/B)$.

Let T, B and $\mathcal{A}$ be as in def. 2.1.2. Let $S(B_n(\Sigma(B)))$ be the stone space of $B_n(\Sigma(B))$, i.e. the set of ultrafilters. Note that the subsets $V_\varphi = \{x \in S(B_n(\Sigma(B))) : [\varphi]_{\mathrm{T}} \in x\}$ for $\varphi \in F_n(\Sigma(B))$ generate the Stone topology. Then if we write $S_n(B/\mathcal{A})$ for the set of all complete n-types of $\mathcal{A}$ over B and let $U_\varphi = \{p \in S_n(B/\mathcal{A}) : \varphi \in p\}$ for each $\varphi \in F_n(\Sigma(B))$, we recover the Stone space and Stone topology in the obvious way: $S(B_n(\Sigma(B))) = \{[p]_{\mathrm{T}} : p \in S_n(B/\mathcal{A})\}$ and $V_\varphi = \{[p]_{\mathrm{T}} : p \in U_\varphi\}$. For this reason we generally call $S_n(B/\mathcal{A})$ the Stone Space as well.

2.2. Saturation.

An important property of structures is saturation, in fact the following elementry result shows that it's so important that Morley's theorem reduces to a statement about saturation. Informally, being λ-saturated means that anything that can be said with less then λ elements about an element of some elementary extension of $\mathcal{A}$ is true of some element in $\mathcal{A}$. Given a Σ-structure $\mathcal{A}$,

Definition 2.2.1. A structure $\mathcal{A}$ is λ-*saturated* if and only if for every subset $B \subseteq |\mathcal{A}|$ of cardinality less than λ, all $p \in S_1(B/\mathcal{A})$ are realized by elements of $\mathcal{A}$. $\mathcal{A}$ is *saturated* if $\mathcal{A}$ is $\#|\mathcal{A}|$-saturated.

Lemma 2.2.2 ([Hod97, Theorem 8.1.8])**.** *If $\mathcal{A} \equiv \mathcal{B}$ and $\mathcal{A}$, $\mathcal{B}$ are of the same cardinality, then if $\mathcal{A}$, $\mathcal{B}$ are both saturated, then $\mathcal{A} \cong \mathcal{B}$.*

Given lemma 2.2.2, proving Morley's theorem reduces to showing that if T is a complete λ-categorical theory for some $\lambda > \omega$, then every uncountable model of T is saturated. Note that if $\mathcal{A}$ is saturated then by induction we have that for all $B \subseteq |\mathcal{A}|$ such that $\#|B| < \#|\mathcal{A}|$, all $p \in S_n(B/\mathcal{A})$ are realized in $\mathcal{A}$, for all n.

[10] A proof that these three conditions are equivalent will be provided in a later draft. Note that (3) is used by Morley to define type, [Mor65, 518] while Hodges uses (1). [Hod97]

2.3. **Stability.** Given lemma 2.2.2, the proof of Morley's theorem follows immediately from two lemmas that will be given below. Before giving them we must define the notion of stability.

As Hodges describes, algebraists often think of algebraically closed fields as "big models" of the field axioms, in the sense that they generally like to think of fields as their images in their algebraic closure. If it weren't for set-theoretic paradoxes model theorists would like to talk about "big models" of any old complete theory T, in the sense that any model of T could be thought of as its image in the big model. [Hod97, 211-2] To borrow a phrase from Hodges, they draw in their horns though and settle for something less:

Definition 2.3.1. Given a signature Σ, a Σ-structure $\mathcal{A}$ is λ-*big* if and only if for every subset $B \subseteq |\mathcal{A}|$ with cardinality less than λ, $(\mathcal{A}; B)$ is such that if Σ' is $\Sigma(B)$ with relation symbol R added, $\mathcal{B}$ a Σ'-structure such that $(\mathcal{A}; \bar{b}) \equiv_{\Sigma(B)} \mathcal{B}$, then we can interpret R by a relation $R^{(\mathcal{A};B)}$ making $(\mathcal{A}; B)$ a Σ'-structure so that $(\mathcal{A}; B) \equiv \mathcal{B}$.

If $\mathcal{A}$ is λ-big, $\mathcal{B}$ another structure of cardinality less than λ and $\mathcal{A} \equiv \mathcal{B}$, then $\mathcal{B} \preccurlyeq \mathcal{A}$. [Hod97, 212] In other words, $\mathcal{A}$ is λ-big if we can view all equivalent structures smaller than λ as substructures of $\mathcal{A}$. As a fact I leave unproved, any algebraically closed field of infinite transcendence degree and characteristic n is λ-big. Hence it will be a "big model" of the axioms for alegrabically closed fields with characteristic n. [Hod97, 217] As should not be hard to imagine, being λ-big implies being λ-saturated. It follows that all uncountable algebraically closed fields are saturated. What makes being λ-big useful is that just like algebraic closures always exist, λ-big extensions of structures always exist. [Hod97, Theorem 8.2.1]

The last important property is stability. Let T be a complete theory and let ϑ be some very large cardinal that's possibly inaccessable. Let $\mathcal{M}$ be a ϑ-big model of T, call it the *monster model* of T.

Definition 2.3.2. For $\lambda < \vartheta$ we say that T is λ-*stable* if for every model $\mathcal{A}$ of T and every set B of at most λ elements of $\mathcal{A}$, $\# S_1(B/\mathcal{M}) \leq \lambda$.

2.4. **The Category $\mathscr{N}$(T).** Note that the notion of a monster model allows us to define a category that will be useful in proving the following lemmas. If T is a complete theory, let $\mathcal{M}$ be a monster model of T. If $\Sigma = \sigma(\mathcal{M})$, define the category of Σ-*Structures over T*, $\mathscr{N}(\mathrm{T}, \mathcal{M})$, by letting:

1. $\mathrm{Ob}(\mathscr{N}(\mathrm{T}, \mathcal{M})) = \{\mathcal{A} : \mathcal{A} \leq \mathcal{M}\}$
2. $\mathrm{Hom}_{\mathscr{N}(\mathrm{T},\mathcal{M})}(\mathcal{A}, \mathcal{B}) = \{f : f : \mathcal{A} \rightarrowtail \mathcal{B}\}$

That is, the objects of $\mathscr{N}(\mathrm{T},\mathcal{M})$ are the substructures of $\mathcal{M}$ while the arrows (morphisms) are the embeddings between the substructures. Note that some structures $\mathcal{A} \in \mathrm{Ob}(\mathscr{N}(\mathrm{T},\mathcal{M}))$ might not be models of T, although since T is complete every one of its models (which is smaller than $\mathcal{M}$) is an object of $\mathscr{N}(\mathrm{T},\mathcal{M})$.[11] Since it normally doesn't matter what $\mathcal{M}$ is choosen we shall write $\mathscr{N}(\mathrm{T})$ instead of $\mathscr{N}(\mathrm{T},\mathcal{M})$.

2.5. **Two Lemmas.** Now the following lemmas connect together the notions of stability and saturation in a way that yields Morley's theorem.

Lemma 2.5.1 ([Hod97, Corollary 9.4.6])**.** *If T is a theory in a countable language $L_{\omega\omega}(\Sigma)$ and is λ-categorical for some $\lambda > \omega$, then T is ω-stable.*

Lemma 2.5.2 ([Mor65, Theorem 5.5])**.** *If T is a theory in a countable language $L_{\omega\omega}(\Sigma)$, is λ-categorical for some $\lambda > \omega$ and is ω-stable, then all uncountable models of T are saturated.*

Lemma 2.5.2 was the crucial lemma proved by Morley in [Mor65], using a generalization of Krull dimension to arbitrary relations on structures now called Morley rank. Lemma 2.5.1 is proved using elementry model-theoretic results of Skolem along with the work of Ehrenfeucht and Mostowski.

3. Structures from Linear Orderings

In order to prove lemma 2.5.1 we are going to assume we have some monster model $\mathcal{M}$ for the theory T mentioned in the antecedent. We can do this because assuming T has no finite models, T being λ-categorical implies that T is complete too. Hence every model of T can be thought of as a substructure for *some* monster model. If the monster model we picked isn't big enough we silently pick a bigger monster model.

3.1. **EM Structures.** The special sort of substructures we want are called Ehrenfeucht-Mostowski structures. They are structures generated from linearly ordered sets, i.e. sets with an ordering relation that is irreflexive, transitive and total, although the linear ordering on the set may have nothing to do with the structure itself. Given a signature Σ, a Σ-structure $\mathcal{A}$, linear ordering η contained in $\mathcal{A}$ whose ordering

[11] For example, let Σ be the signature of fields and T the axioms for algebraically closed fields of infinite transcendence degree with characteristic 0. Then $\{1,0\}$ is a sub-Σ-structure of $\mathbb{C}$ and so $\{1,0\} \in \mathrm{Ob}(\mathscr{N}(\mathrm{T},\mathbb{C}))$, but $\{1,0\}$ is not a model of T

relation need not have anything to do with $\mathcal{A}$, $[\eta]^k$ the set of all finite sequences $a_1, \ldots, a_k$ of $a_i \in \eta$ such that $a_1 <^\eta \ldots <^\eta a_n$, then

Definition 3.1.1. For all complete theories T, an *Ehrenfeucht-Mostowski functor in T*, an EM functor for short, is a functor

$$\mathcal{F} : \mathbf{Ord_L} \to \mathscr{N}(\mathrm{T})$$

that is a functor from the category of linear orderings to the category of Σ-structures over T, such that

(1) For each linear ordering η, $\mathcal{F}(\eta)$ is generated by η.
(2) For each embedding $f : \eta \to \xi$ of linear orderings,

$$\mathcal{F}(f) : \mathcal{F}(\eta) \rightarrowtail \mathcal{F}(\xi)$$

is a structure embedding extending f in the natural way.

Note that since η generates $\mathcal{F}(\eta)$, every element of $\mathcal{F}(\eta)$ is of the form $t^{\mathcal{F}(\eta)}(\bar{a})$ for some $\bar{a} \in [\eta]^k$.

Definition 3.1.2. (1) The *theory of η in $\mathcal{A}$*, $\mathrm{Th}(\mathcal{A}, \eta)$, is the set of all $\varphi(x_1, \ldots, x_k) \in L_{\omega\omega}(\sigma(\mathcal{A}))$ such that $\mathcal{A} \models \varphi(\bar{a})$, $\forall \bar{a} \in [\eta]^k$.
(2) The *theory of the EM functor* $\mathcal{F} : \mathbf{Ord_L} \to \mathscr{N}(\mathrm{T})$, $\mathrm{Th}(\mathcal{F})$, is the set of all $\varphi(x_1, \ldots, x_k) \in L_{\omega\omega}(\sigma(\mathrm{T}))$ such that for every linear ordering η and every $\bar{a} \in [\eta]^k$, $\mathcal{F}(\eta) \models \varphi(\bar{a})$.

Definition 3.1.3. (1) A linear ordering η is *φ-indiscernible* in $\mathcal{A}$ containing η for some $\varphi(\bar{x}) \in L_{\omega\omega}(\Sigma)$ if for any two $\bar{a}, \bar{b} \in [\eta]^k$, $\mathcal{A} \models \varphi(\bar{a})$ iff $\mathcal{A} \models \varphi(\bar{b})$.
(2) A linear ordering is *indiscernible* if it is *φ-indiscernible* for all φ.

3.2. **Sliding.** The following results are important facts about EM models that will be used below. The first result is usually called *sliding*.

Theorem 3.2.1 ([Hod97, Theorem 9.1.1])**.** *If $\mathcal{F}$ is an EM functor in Σ and η, ξ are linear orderings and $\bar{a} \in [\eta]^k$, $\bar{b} \in [\xi]^k$, then for every quantifier-free $\varphi(x_1, \ldots, x_k) \in L_{\omega\omega}(\Sigma)$, $\mathcal{F}(\eta) \models \varphi(\bar{a})$ iff $\mathcal{F}(\xi) \models \varphi(\bar{b})$.*

Proof. Find a linear ordering ζ and embeddings $f : \eta \to \zeta$, $g : \xi \to \zeta$ such that $f(\bar{a}) = g(\bar{b})$. Consider

$$\mathcal{F}(\eta) \overset{\mathcal{F}(f)}{\to} \mathcal{F}(\zeta) \overset{\mathcal{F}(g)}{\to} \mathcal{F}(\xi)$$

It follows almost trivially from the definition of a structure embedding that structure embeddings preserve quantifier-free formula, i.e. given $h : \mathcal{A} \rightarrowtail \mathcal{A}'$ and $\varphi(\bar{x})$ quantifier-free, $\mathcal{A} \models \varphi(\bar{a})$ iff $\mathcal{A}' \models \varphi(h(\bar{a}))$. The theorem follows quickly from the fact that $\mathcal{F}(f)$, $\mathcal{F}(g)$ are structure embeddings. □

Theorem 3.2.2 ([Hod97, Lemma 9.1.3])**.** *If $\mathcal{F}$ is an EM functor in Σ, η an infinite linear ordering, and $\mathcal{F}(\eta) \models \varphi$ for a formula φ built up from quantifier-free formulas by means of $\bigwedge$, $\bigvee$ and universal quantification at most, then $\varphi \in Th(\mathcal{F})$.*

Proof. Assume φ is $\forall \bar{x} \psi(\bar{x})$ with ψ quantifier-free. Let ζ be any linear ordering and $\bar{a}$ a sequence of elements from $\mathcal{F}(\zeta)$. Since ζ generates $\mathcal{F}(\zeta)$, there is some finite subordering ζ_0 of ζ such that $\bar{a}$ is in $\mathcal{F}(\zeta_0)$. Since η is infinite, there is an embedding $f : \zeta_0 \to \eta$. Since $\mathcal{F}(\eta) \models \forall \bar{x} \psi(\bar{x})$, $\mathcal{F}(\eta) \models \psi(\mathcal{F}(f)(\bar{a}))$, so $\mathcal{F}(\zeta_0) \models \varphi(\bar{a})$, since $\varphi(\bar{x})$ was quantifier-free. Hence $\mathcal{F}(\zeta) \models \psi(\bar{a})$. Since ζ and $\bar{a}$ were arbitrary, $\psi(\bar{x}) \in \mathrm{Th}(\mathcal{F})$, so it trivially follows that $\varphi \in \mathrm{Th}(\mathcal{F})$. Since all φ built in the way described can be put in this form, this completes the proof. ☐

3.3. **A Sufficient Condition for EM Functors.** Here we describe a sufficient condition on complete theories T for the existance of an EM functor $\mathcal{F} : \mathbf{Ord_L} \to \mathscr{N}(\mathrm{T})$.

Theorem 3.3.1. *Let $\mathcal{A}$ be a Σ-structure containing ω as generators. If $Th(\Sigma, \mathcal{A})$ is a Skolem theory and ω is φ-indiscernible in $\mathcal{A}$ for all atomic formula φ of $L_{\omega\omega}(\Sigma)$, then there exists an EM functor*

$$\mathcal{F} : \mathbf{Ord_L} \to \mathscr{N}(Th(\Sigma, \mathcal{A}))$$

such that $\mathcal{A} \cong \mathcal{F}(\omega)$.[12]

Proof. Note that the proof will be provided in a later draft. The machinary developed in sections 3.1 and 3.2 is used in the proof. ☐

This result essentially says that any Skolem theory T with an infinite model has models that are generated by indiscernible sequences η, for all linear orders η.

3.4. **Proof of Lemma 2.5.1.** Now we come to the proof of lemma 2.5.1. We will need one more lemma.

Theorem 3.4.1 ([Hod97, Theorem 9.1.7.b], Thinning)**.** *Given a Skolem theory T in $L_{\omega\omega}(\Sigma)$ and $\mathcal{A}$ an EM model of T, if η for $\mathcal{A} = \mathcal{F}(\eta)$ is not only a linear ordering but also is well ordered, i.e. each subset has a*

[12]Note that this functor is unique up to a natural isomorphism on functors. If we did not require EM functors $\mathcal{F}$ to send linear orders into $\mathscr{N}(\mathrm{T})$ this would be what Hodges calls the *stretching* theorem [Hod97, Theorem 9.1.4]. But since we do require it, this theorem also subsumes the Ehrenfeucht-Mostowski theorem [Hod97, Theorem 9.1.6].

smallest element, and B is a subset of $\mathcal{F}(\eta)$, then the number of complete 1-types over B that are realized in $\mathcal{A}$ is at most $\#L_{\omega\omega}(\Sigma) + \#B$.

Proof. The well ordering condition on η is equivalent to assuming that η is a ordinal, so assume η is a ordinal. Note that any element $\beta \in B$ is of the form $t_\beta^{\mathcal{F}(\eta)}(\bar{b}_\beta)$ for some term $t_\beta(\bar{x})$ of $L_{\omega\omega}(\Sigma)$ and increasing sequence $\bar{b}_\beta$ from η. Let W be the smallest subset of η such that all $\bar{b}_\beta$, $\beta \in B$ lie in W. So $\#W \leq \#B + \omega$. Now let $t'(\bar{y})$ be any term of $L_{\omega\omega}(\Sigma)$. Since η is an indiscernible sequence for $\mathcal{F}(\eta)$, which follows from theorem 3.2.1 and lemma 3.2.2, for each increasing $\bar{c}$ from η the type of the elements $t'^{\mathcal{F}(\eta)}(\bar{c})$ over B is completely determined by the positions of the elements of $\bar{c}$ relative to the elements of W in η. Since η is well-ordered, there are at most $\#W + \omega$ ways that $\bar{c}$ can lie relative to W. So the elements $t'^{\mathcal{F}(\eta)}(\bar{c})$ with $\bar{c}$ increasing in η account for at most $\#W + \omega$ complete types over B. Since there are at most $\#L_{\omega\omega}(\Sigma)$ terms $t'(\bar{y})$, yielding a total of $\#W + \#L_{\omega\omega}(\Sigma) = \#B + \#L_{\omega\omega}(\Sigma)$ types of elements over B.[13] □

Proof of lemma 2.5.1. Let T be a theory in a countable language $L_{\omega\omega}(\Sigma)$ and be λ-categorical for some $\lambda > \omega$. If T is not already a Skolem theory we Skolemize T to a theory T^+ in a countable language $L_{\omega\omega}(\Sigma^+)$. By theorem 3.3.1, there is an EM model $\mathcal{F}(\lambda)$ of T^+. Let X be any countable set of elements of $\mathcal{F}(\lambda)$. Since λ is well-ordered, the thinning theorem tells us that at most countably many complete 1-types over X are realized in $\mathcal{F}(\lambda)$. Hence the same is true in any Σ-structure $\mathcal{A}$ such that $\mathcal{F}(\lambda) \cong_\Sigma \mathcal{A}$.

Now suppose that T is *not* ω-stable. Let Y be a countable set of elements of the monster model $\mathcal{M}$ such that $S_1(Y/\mathcal{M})$ is uncountable. By the downward Löwenheim-Skolem, there exists an $\mathcal{A}' \subseteq \mathcal{M}$ with cardinality λ that contains Y and elements realizing uncountably many of the types in $S_1(Y\mathcal{M})$. But λ-categoricity implies that $\mathcal{A}'$ is isomorphic to $\mathcal{A}$, but only countably many complete 1-types are realized in $\mathcal{A}$, so T is ω-stable. □

4. Dimension Theory

The goal of this section is to canvas the proof of lemma 2.5.2. The main tool needed for the proof is the dimension theory that Morley built around the Stone spaces of 1-types.

[13] Second half of the proof taken directly from [Hod97, 256].

4.1. **Stone Spaces as Contravariant Functors.** Let T be a complete theory and let $\mathcal{M}$ be a monster model of T. Through out this section we will "work in" $\mathcal{M}$ without reference to it, or rather we will "work in" $\mathscr{N}(\mathrm{T})$. Hence all Stone spaces of complete 1-types w.r.t. X, written $S_1(X)$, are Stone spaces of complete 1-types *of* $\mathcal{M}$ w.r.t X, i.e. $S_1(X) := S_1(X/\mathcal{M})$, and X is assumed to be the domain of some substructure of $\mathcal{M}$, i.e. $X = |\mathcal{A}|$ where $\mathcal{A} \leq \mathcal{M}$. Thus we want to view the Stone space construction as a contravariant functor from the category of Σ-structures over T to the category of topological spaces.

Given an embedding $f : \mathcal{A} \rightarrowtail \mathcal{B}$ (again assuming that $\mathcal{A}, \mathcal{B} \leq \mathcal{M}$), let $\tilde{f} : F_1(\Sigma(|\mathcal{A}|)) \to F_1(\Sigma(|\mathcal{B}|))$ be defined by letting $\tilde{f}(\psi)$ be the formula obtained by substituting, for each $a \in \mathcal{A}$, the term naming $f(a)$ for each occurance of the term naming a in ψ. Then,

Definition 4.1.1. Let the contravariant functor $S_1 : \mathscr{N}(\mathrm{T}) \to \mathbf{Top}$ be defined so that for $\mathcal{A} \in \mathrm{Ob}(\mathscr{N}(\mathrm{T}))$ and $f \in \mathrm{Hom}_{\mathscr{N}(\mathrm{T})}(\mathcal{A}, \mathcal{B})$,

$$\mathcal{A} \longmapsto S_1(|\mathcal{A}|)$$

$$f : \mathcal{A} \rightarrowtail \mathcal{B} \longmapsto S_1(f) : S_1(|\mathcal{B}|) \to S_1(|\mathcal{A}|)$$

where $S_1(f)$ is defined so that $p \longmapsto \tilde{f}^{-1}(p)$. We will aberviate $S_1(f)$ with f^*.

Note that f^* is continuous, $f^{*-1}(U_\varphi) = U_{\tilde{f}(\varphi)}$ and f^* surjective. If $|\mathcal{A}| \subseteq |\mathcal{B}|$ and $i : |\mathcal{A}| \hookrightarrow |\mathcal{B}|$ is the inclusion map and $p \in S_1(|\mathcal{B}|)$, then $i^*(p) = p \cap S_1(|\mathcal{A}|)$. The image of $\mathscr{N}(\mathrm{T})$ in **Top** under S_1 is a category and is the dual to $\mathscr{N}(\mathrm{T})$. Following Morley we denote it by $\mathscr{C}(\mathrm{T})$.

4.2. **Rank.** Now we come to the crucial definition. Given a stone topology $S_1(X)$ (where again $X = |\mathcal{A}|$ for some $\mathcal{A} \leq \mathcal{M}$), we define subtopologies $S_1^\kappa(X)$ and $\mathrm{Tr}^\kappa(X)$ as follows.

Definition 4.2.1. For each ordinal κ,

(1) $S_1^\kappa(X) = S_1(X) - \bigcup_{\lambda<\kappa} \mathrm{Tr}^\lambda(X)$.

(2) $p \in \mathrm{Tr}^\kappa(X)$ if and only if
   (a) $p \in S_1^\kappa(X)$, and
   (b) for all $\mathcal{B} \in \mathrm{Ob}(\mathscr{N}(\mathrm{T}))$ and all $f^* \in \mathrm{Hom}_{\mathscr{C}(\mathrm{T})}(S_1(|\mathcal{B}|), S_1(X))$, $f^{*-1}(p) \cap S_1^\kappa(|\mathcal{B}|)$ is a set of isolated points in $S_1^\kappa(|\mathcal{B}|)$.[14]

[14]A point $p \in S_1^\kappa(|\mathcal{B}|)$ is *isolated* if $p \in U_\varphi \subseteq Z$ for some $Z \subseteq S_1^\kappa(|\mathcal{B}|)$ and $\varphi \in F_1(\Sigma(|\mathcal{B}|))$ such that no other points p' of $S_1^\kappa(|\mathcal{B}|)$ are in Z. Consider the case when $\kappa = 0$, then $S_1^\kappa(\mathcal{A}) = S_1(\mathcal{A})$. From a logical point of view, an isolated point of $S_1(|\mathcal{A}|)$, i.e. of $S_1^0(|\mathcal{A}|)$, is a set p of formulas from $L_{\omega\omega}(\Sigma(\mathcal{A}))$ with one free variable that is maximally consistent with $\mathrm{Th}(\Sigma, \mathcal{A})$ and such that each $\varphi \in p$ is inconsistent with any other such formula not in p.

Definition 4.2.2. $p \in S_1(X)$ is *algebraic* if $p \in \mathrm{Tr}^0(X)$; p is *transcendental in rank* κ if $p \in \mathrm{Tr}^\kappa(X)$.

The contemporary notion of Morley rank, which assigns a rank $RM(\varphi)$ to each formula $\varphi \in F_n(\Sigma)$, was first introduced and interpreted in terms of Morley's work by Lachlan in [Lac71]. For each $\varphi \in F_n(\Sigma)$, $RM(\varphi)$ is either -1, an ordinal or ∞. One can see a text like [Hod97, 265] for a definition of $RM(\varphi)$ not in terms of $\mathrm{Tr}^\kappa(\mathcal{A})$. In the case when $\varphi \in F_1(\Sigma(|\mathcal{A}|))$, the relationship between $\mathrm{Tr}^\kappa(|\mathcal{A}|)$ and $RM(\varphi)$ is as follows, [Bal73, 40]

$$RM(\varphi) = \begin{cases} -1 & \text{if } \varphi(\mathcal{A}) = \varnothing \\ \sup\{\kappa : \exists p \in U_\varphi, p \in \mathrm{Tr}^\kappa(|\mathcal{A}|)\} & \text{otherwise} \end{cases}$$

Generalizing the definition of $S_1^\kappa(|\mathcal{A}|)$ to a subspace $S_n^\kappa(|\mathcal{A}|)$, $RM(\varphi)$ can be defined in a similar way for all $\varphi \in L_{\omega\omega}(\Sigma(|\mathcal{A}|))$.

It can be proven that if $\varphi(\mathcal{M}^n) = \psi(\mathcal{M}^n)$, then $RM(\varphi) = RM(\psi)$. Hence Morley rank can be assigned to all definable sets $X \subseteq \mathcal{M}^n$ so that $RM(X) = RM(\varphi)$ for $X = \varphi(\mathcal{M}^n)$. [Hod97, Lemma 9.3.2]

For an example, assume that $\mathcal{M}$ is some algebraically closed field of char$= 0$ (and hence a monster model for the field axioms, which is, of course, a theory we'll denote by T_f.) If $X \subseteq \mathcal{M}^n$ is an irreducible algebraic set, i.e. an algebraic variety, then $RM(X)$ is the Krull dimension of X. [Hod97, 264] This fact isn't surprising considering the definition of $RM(\varphi)$ in terms of $\mathrm{Tr}^\kappa(\mathcal{A})$ and the fact that complete types are, roughly, dual prime ideals of $B_n(\Sigma(|\mathcal{A}|))$. If $\mathcal{M} = \mathbb{C}$ and $\varphi \in L_{\omega\omega}(\sigma(\mathrm{T}_f)(\mathbb{C}))$ defines a variety V, then $RM(\varphi)$ is the dimension of V as a complex manifold.

4.3. **Totally Transcendental Theories.** Morley proved the following lemma which we leave unproved. From it we define the important notion of a totally transcendental theory. For all complete theories T:

Lemma 4.3.1 ([Mor65, Lemma 2.6])**.**

1. *There is an ordinal $\alpha_T < (2^{\aleph_0})^+$ which is the least ordinal such that for all $\mathcal{A} \in \mathscr{N}(T)$ and all $\kappa > \alpha_T$, $S_1^{\alpha_T}(\mathcal{A}) = S_1^\kappa(\mathcal{A})$.*
2. *Further, if $S_1^{\alpha_T}(\mathcal{A}) = \varnothing$ for some $\mathcal{A} \in \mathscr{N}(T)$, then α_T is not a limit ordinal and for every $\mathcal{B} \in \mathscr{N}(T)$, $S_1^{\alpha_T}(\mathcal{B}) = \varnothing$ and $S_1^\kappa(\mathcal{B}) = \varnothing$ for any $\kappa > \alpha_T$.*

Definition 4.3.2. A theory T is *totally transcendental* if $S_1^{\alpha_T}(\mathcal{A}) = \varnothing$ for some (and hence every) $\mathcal{A} \in \mathscr{N}(\mathrm{T})$. Equivalently, T is totally transcendental if $RM(x = x) < \infty$, for RM defined w.r.t. the monster model $\mathcal{M}$ of T.

By def 4.3.2 and lemma 4.3.1, if T is totally trascendental we have that $S_1^{\kappa}(\mathcal{A}) = \varnothing$ for all $\kappa \geq \alpha_T$, and by def 4.2.1 we have $S_1^{\alpha_T}(\mathcal{A}) = S_1(\mathcal{A}) - \bigcup_{\lambda<\alpha_T} \mathrm{Tr}^{\lambda}(\mathcal{A})$, so $S_1(\mathcal{A}) = \bigcup_{\lambda<\alpha_T} \mathrm{Tr}^{\lambda}(\mathcal{A})$. In other words, every type of a totally transcendental theory is transcendental in some rank λ.

4.4. **Proof of Lemma 2.5.2.** Given this definition, we can state three lemmas from which lemma 2.5.2 quickly follows.

Lemma 4.4.1 ([Mor65, Theorem 2.8])**.** *T is totally transcendental if and only if it is ω-stable.*

Lemma 4.4.2 ([Mor65, Theorem 5.2])**.** *If T is totally transcendental and $\lambda > \omega$, then there is a model $\mathcal{A}$ of T such that $\#|\mathcal{A}| = \lambda$ which is saturated over its countable substructures.*

Lemma 4.4.3 ([Mor65, Theorem 5.4])**.** *If T is totally transcendental and has an uncountable model which is not saturated, then for each $\lambda > \omega$, T has a model of size λ that is not saturated over any of its countable substructures.*

Proofs for these three lemmas will be provided in a later draft. For now, we can give the two most exciting proofs:

Proof of lemma 2.5.2. Let T be a theory in a countable language $L_{\omega\omega}(\Sigma)$, be λ-categorical for some $\lambda > \omega$ and be ω-stable. Assume that not all uncountable models of T are saturated, say there is an uncountable non-saturated model $\mathcal{A}$ of size κ. T is totally transcendental by lemma 4.4.1. By lemma 4.4.3, for all $\delta > \omega$ T has a model of size δ that is not saturated over any of its countable substructures. In particular T has a model of size λ which is not saturdated over any of its countable substructures. But this is impossible, since by lemma 4.4.2, there is a model $\mathcal{B}$ of T of size λ which is saturated over its countable substructures and T is λ-categorical, i.e. $\mathcal{A} \cong \mathcal{B}$. Therefore T cannot have a countable, non-saturated model. □

Proof of Morley's Theorem. It now follows from lemmas 2.5.1, 2.5.2, and 2.2.2 that if $T \subseteq L_{\omega\omega}(\Sigma)$, Σ countable, is λ-categorical for some $\lambda > \omega$, then T is λ-categorical for all $\lambda > \omega$. □

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